Math 211 - Bonus Exercise 13 (please discuss on Forum)

Let Ab be the category whose objects are abelian groups, and whose morphisms are group homomorphisms.

1. Prove that Ab is a category;
2. Prove that there is a functor $I : \mathrm{Ab} \to \mathrm{Gr}$ such that $I(A) = A$;
3. Prove that there is a functor $(-)^{\mathrm{ab}} : \mathrm{Gr} \to \mathrm{Ab}$ such that $G^{\mathrm{ab}} = G/[G, G]$;
4. Prove that $(-)^{\mathrm{ab}}$ and I are adjoint: there exists a natural bijection

$$\mathrm{Mor}_{\mathrm{Ab}}(G^{\mathrm{ab}}, A) \cong \mathrm{Mor}_{\mathrm{Gr}}(G, I(A))$$

for all groups G and abelian groups A.